Period 1, Sept 25, 2024

$$y = (x^2+4x)^2 = x^4+8x^3+16x^2$$

$$\frac{dy}{dx} = 4x^3+24x^2+32x$$

$$y = (x^2+4x)^2 \Rightarrow y = L^2$$

$$L = x^2+4x \qquad \frac{dy}{dL} = 2L$$

$$\frac{dL}{dx} = 2x+4$$

$$\frac{d\cancel{L}}{dx}\cdot\frac{dy}{\cancel{dL}} = (2x+4)(2L) = 2(2x+4)(x^2+4x)$$

$$= (4x+8)(x^2+4x)$$

$$4x^3+8x^2+16x^2+32x$$

$$4x^3+24x^2+32x$$

$$y = (x^2+4x)^{157} \Rightarrow y = L^{157}$$

$$L = x^2+4x \qquad \frac{dy}{dL} = 157L^{156}$$

$$\frac{dL}{dx} = 2x+4$$

$$\frac{dy}{dx} = \frac{d\cancel{L}}{dx}\cdot\frac{dy}{\cancel{dL}} = (2x+4)(157L^{156}) = 157(2x+4)(x^2+4x)^{156}$$

$$y = 3\sin(x^4+7x^2) \Rightarrow y = 3\cdot\sin L$$

$$L = x^4+7x^2 \qquad \frac{dy}{dL} = 3\cos L$$

$$\frac{dL}{dx} = 4x^3+14x$$

$$\frac{dy}{dx} = \frac{\cancel{dL}}{dx}\cdot\frac{dy}{\cancel{dL}} = (4x^3+14x)(3\cos L) = 3(4x^3+14x)\cos(x^4+7x^2)$$

$$y = 3\sin(x^4+7x^2)^{132} \Rightarrow y = 3\sin L^{132} \Rightarrow y = 3\sin u$$

$$L = x^4+7x^2 \qquad u = L^{132} \qquad \frac{dy}{du} = 3\cos u$$

$$\frac{dL}{dx} = 4x^3+14x \qquad \frac{du}{dL} = 132L^{131}$$

$$\frac{\cancel{dL}}{dx}\cdot\frac{\cancel{du}}{\cancel{dL}}\cdot\frac{dy}{\cancel{du}} = \frac{dy}{dx}$$

$$(4x^3+14x)(132L^{131})(3\cos u) = \frac{dy}{dx}$$

$$(4x^3+14x)(132(x^4+7x^2)^{131}\cdot 3\cos(L^{132})) = \frac{dy}{dx}$$

$$(4x^3+14x)(132(x^4+7x^2)^{131}\cdot 3\cos(x^4+7x^2)^{132} = \frac{dy}{dx}$$

$y = 3\sin^7\sqrt{x^4+3x} \Rightarrow y = 3\sin^7\sqrt{L} \Rightarrow y = 3\sin^7 u \Rightarrow y = 3n^7$

$L = x^4+3x$ $\quad u = \sqrt{L} = L^{\frac{1}{2}}$ $\quad n = \sin u$ $\quad \frac{dy}{dn} = 21n^6$

$\frac{dL}{dx} = 4x^3+3$ $\quad \frac{du}{dL} = \frac{1}{2\sqrt{L}}$ $\quad \frac{dn}{du} = \cos u$

$$\frac{\cancel{dL}}{dx}\cdot\frac{\cancel{du}}{\cancel{dL}}\cdot\frac{\cancel{dn}}{\cancel{du}}\cdot\frac{dy}{\cancel{dn}} = \frac{dy}{dx}$$

$$(4x^3+3)\left(\frac{1}{2\sqrt{L}}\right)\left((\cos u)(21n^6)\right) = \frac{dy}{dx}$$

$$(4x^3+3)\left(\frac{1}{2\sqrt{x^4+3x}}\right)\left(\cos\sqrt{x^4+3x}\right)(21\sin^6 u)$$

$$(4x^3+3)\left(\frac{1}{2\sqrt{x^4+3x}}\right)\left(\cos\sqrt{x^4+3x}\right)\cdot\left(21\sin^6\sqrt{x^4+3x}\right) = \frac{dy}{dx}$$

$$21\left(\sin\sqrt{x^4+3x}\right)^6$$

47. $y = \dfrac{3(1-\sin x)}{2\cos x} = \dfrac{3-3\sin x}{2\cos x}$

$$h(x) = \frac{F(x)}{g(x)} \Rightarrow h'(x) = \frac{F'(x)\cdot g(x) - F(x)\cdot g'(x)}{(g(x))^2}$$

$$\frac{dy}{dx} = \frac{(0-3\cdot\cos x)(2\cos x) - (3-3\sin x)(2(-\sin x))}{(2\cos x)^2}$$

$$\frac{dy}{dx} = \frac{-6\cos^2 x - [-6\sin x + 6\sin^2 x]}{4\cos^2 x} = \frac{-6\cos^2 x - 6\sin^2 x + 6\sin x}{4\cos^2 x}$$

$$\frac{dy}{dx} = \frac{-6(\cos^2 x + \sin^2 x - \sin x)}{4\cos^2 x} = \frac{-6}{4}\left(\frac{1-\sin x}{\cos^2 x}\right) = \frac{-3}{2}\left(\frac{1}{\cos^2 x} - \frac{\sin x\cdot 1}{\cos x\cdot\cos x}\right)$$

$$= -\frac{3}{2}\left(\sec^2 x - \tan x\sec x\right)$$

63. $f(x) = (x^3+4x-1)(x-2)$, $(1, -4)$

$F'(x) = (3x^2+4)(x-2) + (x^3+4x-1)(1)$

$F'(1) = (3(1)^2+4)(1-2) + (1^3+4(1)-1)(1)$

$7\cdot -1 + 4\cdot 1 = -7+4 = -3$

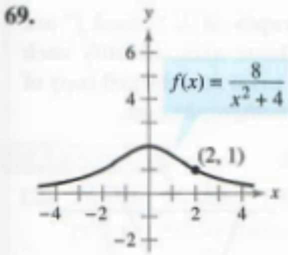
$m = -3$ $\quad$ Point $(1, -4)$

$y-(-4) = -3(x-1)$

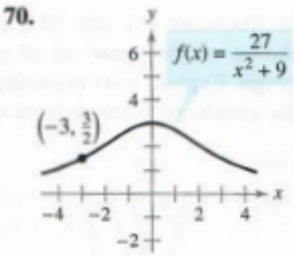
$y+4 = -3x+3 \Rightarrow y = -3x-1$

Famous Curves In Exercises 69–72, find an equation of the tangent line to the graph at the given point. (The graphs in Exercises 69 and 70 are called *Witches of Agnesi*. The graphs in Exercises 71 and 72 are called *serpentines*.)

69.



70.



$$F(x) = \frac{27}{(x^2+9)}$$

$$F'(x) = \frac{0(x^2+9) - 27\cdot 2x}{(x^2+4)^2}$$

$$F'(-3) = \frac{-27\cdot 2\cdot -3}{((-3)^2+9)^2} = \frac{81\cdot 2}{(9\cdot 2)^2} = \frac{\cancel{81}\cdot 2}{\cancel{81}\cdot 4}$$

$$F'(-3) = \frac{1}{2}$$

$(-3, \frac{3}{2})$ $\quad y - \frac{3}{2} = \frac{1}{2}(x-(-3))$

69 $\quad F(x) = \dfrac{8}{x^2+4}$

$$F'(x) = \frac{\cancel{0(x^2+4)} - 8(2x)}{(x^2+4)^2}$$

$$F'(2) = \frac{-16\cdot 2}{(2^2+4)^2} = \frac{-32}{8^2} = \frac{-32}{64} = -\frac{1}{2} = m$$

$y - y_1 = m(x - x_1)$

Point $(2,1)$ $\quad m = -\frac{1}{2}$

$y - 1 = -\frac{1}{2}(x-2)$ $\quad +1 \quad +1$ $\quad \Rightarrow y = -\frac{1}{2}(x-2)+1$

$y = -\frac{1}{2}x + 2$

$x + 2y - 4 = 0$

37. $f(x) = \dfrac{x^2+c^2}{x^2-c^2}$, $\quad c$ is a constant

$$F'(x) = \frac{2x(x^2-c^2) - (x^2+c^2)(2x)}{(x^2-c^2)^2}$$

$$F'(x) = \frac{\cancel{2x^3} - 2c^2x \cancel{-2x^3} - 2c^2x}{(x^2-c^2)^2} = \frac{-4c^2x}{(x^2-c^2)^2}$$

67. $f(x) = \tan x, \quad \left(\frac{\pi}{4}, 1\right)$

$F'(x) = \sec^2 x$

$$F'\left(\tfrac{\pi}{4}\right) = \sec^2\tfrac{\pi}{4} = \frac{1}{\cos^2\frac{\pi}{4}} = \frac{1}{\left(\frac{\sqrt{2}}{2}\right)^2}$$

$$F'\left(\tfrac{\pi}{4}\right) = \frac{1}{\frac{2}{4}} = \frac{1}{1}\cdot\frac{4}{2} = \frac{4}{2} = 2$$

$m = 2$ $\quad$ point $\left(\frac{\pi}{4}, 1\right)$

$y - 1 = 2\left(x - \frac{\pi}{4}\right)$

$y - 1 = 2x - \frac{2\pi}{4}$ $\quad +1 \quad +1$

$y = 2x - \frac{\pi}{2} + 1$

25. $f(x) = \dfrac{4 - 3x - x^2}{x^2 - 1}$

$-(-(x^2+3x-4)(2x))$

$$F'(x) = \frac{(0-3-2x)(x^2-1) - (4-3x-x^2)(2x)}{(x^2-1)^2}$$

$(x-1)(x+4)$

$$F'(x) = \frac{-1(2x+3)(x-1)(x+1) + (x^2+3x-4)(2x)}{[(x-1)(x+1)]^2}$$

$$F'(x) = \frac{\cancel{(x-1)}\,[-1(2x+3)(x+1) + (x+4)(2x)]}{(x-1)^{\cancel{2}}(x+1)^2}$$

In Exercises 59–62, evaluate the derivative of the function at the given point. Use a graphing utility to verify your result.

Function	Point
59. $y = \frac{1 + \csc x}{1 - \csc x}$	$\left(\frac{\pi}{6}, -3\right)$
60. $f(x) = \tan x \cot x$	$(1, 1)$
61. $h(t) = \frac{\sec t}{t}$	$\left(\pi, -\frac{1}{\pi}\right)$
62. $f(x) = \sin x(\sin x + \cos x)$	$\left(\frac{\pi}{4}, 1\right)$

(59)

$$\frac{dy}{dx} = \frac{(0 - \cot x \csc x)(1 - \csc x) - (1 + \csc x)\cdot(+\csc x \cot x)}{(1 - \csc x)^2}$$

$$\frac{dy}{dx} = \frac{-\cot x \csc x + \cancel{\cot x \csc^2 x} - \cot x \csc x - \cancel{\cot x \csc^2 x}}{(1 - \csc x)^2}$$

$$\frac{dy}{dx} = \frac{-2 \cot x \csc x}{(1 - \csc x)^2}$$

$\csc \frac{\pi}{6} = 2$

$\cot \frac{\pi}{6} = \sqrt{3}$

60. $F(x) = \tan x \cot x$

$= \frac{\sin x}{\cos x} \cdot \frac{\cos x}{\sin x}$

$F(x) = 1$

$F'(1) = 0$

$F(x) = \tan x \cot x$

$F'(x) = \sec^2 x \cdot \cot x + \tan x \cdot -\csc^2 x$

$\frac{1}{\cos^2 x} \cdot \frac{\cancel{\cos x}}{\sin x} - \frac{\cancel{\sin x}}{\cos x} \cdot \frac{1}{\sin^2 x} = \frac{1}{\cos x \sin x} - \frac{1}{\cos x \sin x} = 0$

(61) $\frac{\sec T}{T} = h(T)$

$h'(T) = \frac{\sec T \tan T (T) - \sec T (1)}{T^2}$

$h'(T) = \frac{\sec T [\tan T) \cdot T - 1]}{T^2} = \frac{-1[0 \cdot \pi - 1]}{\pi^2}$

$\tan \pi = 0$

$\sec \pi = -1$

$m = \frac{1}{\pi^2}$

45. $g(t) = \sqrt[4]{t} + 6 \csc t$

$g(T) = T^{\frac{1}{4}} + 6 \csc T$

$g'(T) = \frac{1}{4} \cdot T^{\frac{1}{4} - 1 = \frac{-3}{4}} + 6 \cdot -\csc T \cot T$

$g'(T) = \frac{1}{4\sqrt[4]{T^3}} - 6 \csc T \cot T$

49. $y = -\csc x - \sin x$

$\frac{dy}{dx} = -(-\csc x \cot x) - (\cos x)$

$= \frac{\cos x}{\sin^2 x} - \frac{\cos x}{1}$

27. $f(x) = x\left(1 - \frac{4}{x+3}\right)$

$F(x) = x\left(\frac{x+3}{x+3} - \frac{4}{x+3}\right)$

$F(x) = x\left(\frac{x+3-4}{x+3}\right)$

$F(x) = x\left(\frac{x-1}{x+3}\right)$

$x+3-x+1$

$F'(x) = 1\left(\frac{x-1}{x+3}\right) + x\left(\frac{1(x+3) - (x-1)(1)}{(x+3)^2}\right)$

$\frac{x-1}{x+3} + \frac{x(4)}{(x+3)^2}$

$F(x) = \frac{x^2 - x}{x+3}$

$F'(x) = \frac{(2x-1)(x+3) - (x^2-x)(1)}{(x+3)^2}$

28. $f(x) = x^4\left(1 - \frac{2}{x+1}\right)$

$F(x) = x^4\left(\frac{x+1}{x+1} - \frac{2}{x+1}\right) = x^4\left(\frac{x+1-2}{x+1}\right) = \frac{x^4(x-1)}{x+1}$

$F(x) = \frac{x^5 - x^4}{x+1}$

$F'(x) = \frac{(5x^4 - 4x^3)(x+1) - (x^5 - x^4) \cdot 1}{(x+1)^2}$

33. $f(x) = \frac{2 - \frac{1}{x}}{x-3}$

$F(x) = \frac{\frac{2x}{x} - \frac{1}{x}}{x-3} = \frac{\frac{2x-1}{x}}{\frac{x-3}{1}}$

$F(x) = \frac{2x-1}{x} \cdot \frac{1}{x-3} = \frac{2x-1}{x^2-3x}$

$F'(x) = \frac{2(x^2-3x) - (2x-1)(2x-3)}{(x^2-3x)^2}$

34. $g(x) = x^2\left(\frac{2}{x} - \frac{1}{x+1}\right)$

$g(x) = x^2\left(\frac{2(x+1)}{x(x+1)} - \frac{1x}{(x+1)x}\right)$

$g(x) = x^2\left(\frac{2x+2-x}{x(x+1)}\right) = x^{\cancel{2}}\left(\frac{x+2}{\cancel{x}(x+1)}\right) = \frac{x^2+2x}{x+1}$

$g'(x) = \frac{(2x+2)(x+1) - (x^2+2x)(1)}{(x+1)^2} = \frac{2x^2 \cancel{+2x} + 2x + 2 - x^2 \cancel{-2x}}{(x+1)^2}$

$g'(x) = \frac{x^2+2x+2}{(x+1)^2}$

9. $h(x) = \dfrac{\sqrt{x}}{x^3+1} = \dfrac{x^{\frac{1}{2}}}{x^3+1}$

$$h'(x) = \frac{\frac{1}{2\sqrt{x}}(x^3+1) - \sqrt{x}(3x^2)}{(x^3+1)^2}$$

$$h'(x) = \frac{x^3}{2\sqrt{x}} + \frac{1}{2\sqrt{x}} - 3x^{2+\frac{1}{2}}$$

$$h'(x) = \frac{1}{2}x^{3-\frac{1}{2}} + \frac{1}{2\sqrt{x}} - 3x^{2+\frac{1}{2}}$$

$$h'(x) = \frac{1}{2}x^{2\frac{1}{2}} - 3x^{2\frac{1}{2}} + \frac{1}{2\sqrt{x}}$$

$$= -2\tfrac{1}{2} \cdot x^{2\frac{1}{2}} + \frac{1}{2\sqrt{x}}$$

$$\boxed{h'(x) = \frac{-\frac{5}{2} \cdot x^2\sqrt{x} + \frac{1}{2\sqrt{x}}}{(x^3+1)^2}}$$

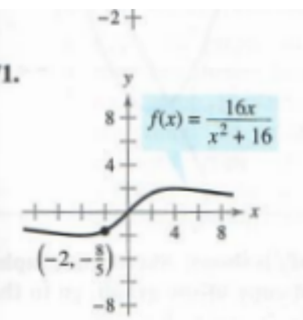
10. $h(s) = \dfrac{s}{\sqrt{s}-1}$

$$h'(s) = \frac{1(\sqrt{s}-1) - s\left(\frac{1}{2\sqrt{s}}\right)}{(\sqrt{s}-1)^2}$$

$$h'(s) = \frac{\sqrt{s} - 1 - \frac{s}{2\sqrt{s}}}{(\sqrt{s}-1)^2} = \frac{\sqrt{s} - 1 - \frac{1}{2}\sqrt{s}}{(\sqrt{s}-1)^2}$$

$$h'(s) = \frac{\frac{1}{2}\sqrt{s} - 1}{(\sqrt{s}-1)^2} = \frac{\sqrt{s}-2}{2(\sqrt{s}-1)^2}$$

71.



$$F'(x) = \frac{16(x^2+16) - 16x(2x)}{(x^2+16)^2}$$

$$F'(x) = \frac{16x^2 + 256 - 32x^2}{(x^2+16)^2}$$

$$F'(x) = \frac{-16x^2+256}{(x^2+16)^2}$$

$$F'(-2) = \frac{-16(4)+256}{(4+16)^2} = \frac{-64+256}{20^2}$$

$$F'(-2) = \frac{192}{400} = \frac{48}{100} = \frac{12}{25} = m$$

$m = \dfrac{12}{25}$ $\left(-2, -\dfrac{8}{5}\right)$

$$y - \left(-\frac{8}{5}\right) = \frac{12}{25}\left(x - (-2)\right)$$

Example 1/2: Identify the inside and outside function in each of the following composite functions. Then find derivative

a) $f(x) = \cos(5x^2)$ $\Rightarrow y = \cos u$

$u = 5x^2$

$\frac{du}{dx} = 10x$

$\frac{dy}{du} = -\sin u$

$\frac{\cancel{du}}{dx} \cdot \frac{dy}{\cancel{du}} = (10x)(-\sin 5x^2)$

b) $g(x) = \sqrt{3x+2}$

$y = \sqrt{3x+2} \Rightarrow y = \sqrt{u} = u^{\frac{1}{2}}$

$u = 3x+2$

$\frac{dy}{du} = \frac{1}{2} \cdot u^{-\frac{1}{2}} = \frac{1}{2\sqrt{u}}$

$\frac{du}{dx} = 3$

$\frac{\cancel{du}}{dx} \cdot \frac{dy}{\cancel{du}} = 3 \cdot \frac{1}{2\sqrt{3x+2}} = \frac{3}{2\sqrt{3x+2}}$

c) $f(x) = \sin^2 x$

d) $g(x) = \frac{1}{5x+7}$

Example 3: Find the derivative of $f(x) = \sqrt[3]{\left(x^2 - 1\right)^2}$

$y = \sqrt[3]{(x^2-1)^2} \Rightarrow y = u^{2/3}$

$u = x^2 - 1$

$\frac{dy}{du} = \frac{2}{3} u^{-\frac{1}{3}} = \frac{2}{3\sqrt[3]{u}}$

$\frac{du}{dx} = 2x$

$F(x) = e^{5x^4 - 3x^2}$

$F'(x) = e^{5x^4 - 3x^2}(20x^3 - 6x)$

Example 9: Find the derivative of $f(x) = x^2\sqrt{1-x^2}$

$$F'(x) = 2x\sqrt{1-x^2} + x^2 \frac{d}{dx}\left(\sqrt{1-x^2}\right)$$

$$F'(x) = 2x\sqrt{1-x^2} + x^2 \cdot \left(\frac{-x}{\sqrt{1-x^2}}\right)$$

$y = \sqrt{1-x^2} \Rightarrow y = \sqrt{u} = u^{\frac{1}{2}}$

$u = 1-x^2$

$\frac{dy}{du} = \frac{1}{2}u^{\frac{1}{2}-1} = \frac{1}{2\sqrt{u}}$

$\frac{du}{dx} = -2x$

$$\frac{d\cancel{u}}{dx} \cdot \frac{dy}{\cancel{du}} = -\cancel{2}x \cdot \frac{1}{\cancel{2}\sqrt{u}} = \boxed{\frac{-x}{\sqrt{1-x^2}}}$$